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Separation of variables for the A_3 elliptic Calogero–Moser system

V V Mangazeev

Centre for Mathematics and its Applications, School of Mathematical Sciences, The Australian National University, ACT0200, Australia

E-mail: vladimir@maths.anu.edu.au

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Abstract

We consider the classical elliptic Calogero–Moser model. A set of canonical separated variables for this model has been constructed in Kuznetsov *et al*. However, the generating function of the separating canonical transform is known only for two- and three-particle cases. We construct this generating function for the next A_3 case as the limit of the conjectured form of the quantum separating operator. We show explicitly that this generating function gives a canonical transform from the set of original variables to the separated ones.

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1. Introduction

The separation of variables (SoV) method is one of the powerful approaches to solve spectral problems for quantum integrable systems (see [2] for an overview). This method has been successfully applied to many integrable systems. However, it appears that the Calogero–Moser system (CMS) [3, 4] (and its relativistic analogue the Ruijsenaars model [5]) is an example where SoV encounters some difficulties. Namely, as shown in [6] the classical r-matrix for the CMS depends explicitly on dynamical variables when a quantization procedure is not known. As a result all approaches concerning the quantum SoV for these systems have used, in fact, *ad hoc* methods. The interest in producing a SoV for the CMS is twofold. The first reason is obvious: the SoV method can help to reduce the multi-dimensional spectral problem for the CMS to a set of one-dimensional ones which are easier to handle. The second reason is a connection of different limits of the CMS with symmetric functions [7]. In particular, the SoV method for the A_2 quantum CMS in the trigonometric limit produces a new integral representation for the A_2 Jack polynomials [8].

In the paper [1] a set of separated canonical variables has been constructed for the classical Ruijsenaars model (which gives CMS when $\lambda \to 0$). The new canonical separated variables come as poles of the properly normalized Baker–Akhiezer function. However, to describe explicitly a transformation to the new set of canonical variables we need to know a generating function. This function was constructed in [1] for the A_2 case. For the A_n, $n > 3$ it satisfies complicated nonlinear partial differential equations (PDEs) and there is little hope of solving them directly. In this paper we will show how to parametrize solutions of this PDE for the A_3 case. This parametrization comes naturally from the asymptotics of the solutions of special systems of linear equations for the separating kernel in the quantum case.

The paper is organized as follows. In section 2 we recall the main properties of the classical CMS and give necessary definitions of Weierstrass elliptic functions. In section 3 we recall the main facts about the SoV [1] for the CMS and introduce some convenient notations. In section 4 we formulate the quantum version of the model and make a conjecture on the quantum separation operator. In section 5 we prove the main theorem that the A_3 generating function is given by the asymptotics of this operator. In section 6 we give some concluding remarks.

2. The classical Calogero–Moser system

The elliptic N-particle CMS [3, 4] is described in terms of canonical variables p_i, q_i, $i = 1, \dots, N$ with Poisson brackets

$$\{p_i, p_j\} = \{q_i, q_j\} = 0 \qquad \{p_i, q_j\} = \delta_{ij} \tag{2.1}$$

and Hamiltonian

$$\mathcal{H} = \sum_{i=1}^{N} p_i^2 + g^2 \sum_{i\neq j} \wp(q_i - q_j) \tag{2.2}$$

where $\wp(x)$ is the Weierstrass function with periods $2\omega_1$ and $2\omega_2$ (see, e.g., [9]).

Let us summarize some important properties of Weierstrass functions [9] to be used later. We define the Weierstrass σ-function by the infinite product

$$\sigma(x) = x \prod_{m,n\neq 0} \left(1 - \frac{x}{\omega_{mn}}\right) \exp\left[\frac{x}{\omega_{mn}} + \frac{1}{2}\left(\frac{x}{\omega_{mn}}\right)^2\right] \tag{2.3}$$

where $\omega_{mn} = 2m\omega_1 + 2n\omega_2$, $m, n \in Z$ and $\Gamma = 2\omega_1 Z + 2\omega_2 Z$ is the period lattice. Then ζ and $\wp$ Weierstrass functions are defined as

$$\zeta(x) = \frac{\sigma'(x)}{\sigma(x)} \qquad \wp(x) = -\zeta'(x). \tag{2.4}$$

The function $\wp(x)$ is an elliptic function of periods $2\omega_1$, $2\omega_2$, which is even and has the only double pole at $z = 0$ in the primitive domain $\mathfrak{D} := \{z = 2\omega_1 x + 2\omega_2 y | x, y \in [0, 1)\}$. The functions $\zeta(x)$ and $\sigma(x)$ are odd functions, which are quasi-periodic, obeying

$$\zeta(x + 2\omega_{1,2}) = \zeta(x) + 2\eta_{1,2} \qquad \sigma(x + 2\omega_{1,2}) = -\sigma(x)\mathrm{e}^{2\eta_{1,2}(x+\omega_{1,2})} \tag{2.5}$$

where $\eta_{1,2} = \zeta(\omega_{1,2})$ and $\eta_1\omega_2 - \eta_2\omega_1 = \frac{\mathrm{i}\pi}{2}$.

They have the following expansions near the origin:

$$\begin{aligned}\sigma(x) &= x - \frac{g_2 x^5}{240} - \frac{g_3 x^7}{840} - \frac{g_2^2 x^9}{161\,280} + \cdots \\ \zeta(x) &= \frac{1}{x} - \frac{g_2 x^3}{60} - \frac{g_3 x^5}{140} - \frac{g_2^2 x^7}{8400} + \cdots\end{aligned} \tag{2.6}$$

with

$$g_2 = 60 \sum_{m,n\neq 0} \frac{1}{\omega_{mn}^4} \qquad g_3 = 60 \sum_{m,n\neq 0} \frac{1}{\omega_{mn}^6}. \tag{2.7}$$

Weierstrass functions satisfy addition theorems; the most important are

$$\zeta(x+y) = \zeta(x) + \zeta(y) + \frac{1}{2}\frac{\wp'(x) - \wp'(y)}{\wp(x) - \wp(y)} \tag{2.8}$$

$$\wp(x+y) + \wp(x) + \wp(y) = [\zeta(x+y) - \zeta(x) - \zeta(y)]^2 \tag{2.9}$$

$$\sigma(x+y)\sigma(x-y) = -\sigma^2(x)\sigma^2(y)[\wp(x) - \wp(y)] \tag{2.10}$$

$$\Phi(u,x)\Phi(u,y) = \Phi(u,x+y)[\zeta(u) + \zeta(x) + \zeta(y) - \zeta(u+x+y)] \tag{2.11}$$

where

$$\Phi(u,x) = \frac{\sigma(u+x)}{\sigma(u)\sigma(x)} \tag{2.12}$$

and the generalized Cauchy identity

$$\det\left[\Phi(u, x_i - y_j)\right] = \Phi(u,\Sigma)\sigma(u,\Sigma)\frac{\prod_{k<l}\sigma(x_k - x_l)\sigma(y_l - y_k)}{\prod_{k,l}\sigma(x_k - y_l)} \tag{2.13}$$

with $\Sigma = \sum_i (x_i - y_i)$.

The system with Hamiltonian (2.2) is completely integrable [3,4,10] and the complete set of integrals of motion can be represented as spectral invariants of the Lax operator. Namely, define the $N \times N$ Krichever Lax operator [11]

$$\mathcal{L}(u) = \sum_{i=1}^{N} p_i E_{ii} - \mathrm{i}g \sum_{i\neq j} \Phi(u, x_i - x_j) E_{ij} \tag{2.14}$$

with matrix E_{ij} having the nonzero entries $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and $\Phi(u,x)$ defined by (2.12). Then a decomposition of $\det(z \cdot \mathbf{1} - \mathcal{L}(u))$ in z

$$\det(z\cdot\mathbf{1} - \mathcal{L}(u)) = \sum_{i=0}^{N} (-1)^i z^{N-i} t_i(u) \tag{2.15}$$

generates a set of commuting Hamiltonians H_i, $i = 1,\ldots,N$ with respect to the Poisson bracket

$$\{H_i, H_j\} = 0 \qquad i,j = 1,\ldots,N. \tag{2.16}$$

3. The separation of variables

In this section we briefly recall the results from sections 3 and 6 of [1] (see also [2]).

We are looking for a canonical transformation K which maps $(q,p) \mapsto (u,v)$, $H_i(x,p) \mapsto H_i(u,v)$ such that there exist N relations

$$\Phi_i(u_i, v_i; H_1,\ldots,H_N) = 0 \qquad i = 1,\ldots,N. \tag{3.1}$$

The main problem is to construct a generating function $\mathcal{F}(u|q)$ which performs such a separation.

A Baker–Akhiezer function is defined as the eigenvector of the Lax operator $\mathcal{L}(u)$

$$\mathcal{L}(u)\Omega(u) = v(u)\Omega(u) \tag{3.2}$$

with a normalization fixed by a linear condition

$$\vec{\alpha}\cdot\Omega \equiv \sum_{i=1}^{N}\alpha_i(u)\Omega_i(u) = 1. \tag{3.3}$$

The separated variables u_i are thought of as *poles of the properly normalized* Baker–Akhiezer function. Then the canonically conjugated variables v_i are the corresponding eigenvalues of $\mathcal{L}(u_i)$ and satisfy separation equations (3.1)

$$\Phi_i \equiv \det(v_i \cdot \mathbf{1} - \mathcal{L}(u_i)) = 0. \tag{3.4}$$

From (3.2), (3.3) it follows that

$$\Omega(u) = \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\mathcal{L}(u) \\ \vdots \\ \vec{\alpha}\mathcal{L}^{n-1}(u) \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 \\ v \\ \vdots \\ v^{n-1} \end{pmatrix}. \tag{3.5}$$

Define the function

$$\mathfrak{B}(u) = \det \begin{pmatrix} \vec{\alpha} \\ \vec{\alpha}\mathcal{L}(u) \\ \vdots \\ \vec{\alpha}\mathcal{L}^{n-1}(u) \end{pmatrix}. \tag{3.6}$$

Then the poles u_i (or separated variables) of the Baker–Akhiezer function are defined from the condition $\mathfrak{B}(u_j) = 0$.

It has been shown in [1] that the simplest normalization condition $\vec{\alpha}(u) = (0, 0, \ldots, 0, 1)$ works for the CMS. With such a normalization the expression for $\mathfrak{B}(u)$ takes the form

$$\mathfrak{B}(u) = \det \begin{pmatrix} 0 & \cdots & 1 \\ \mathcal{L}_{n1} & \cdots & \mathcal{L}_{nn} \\ \vdots & \ddots & \vdots \\ (\mathcal{L}^{n-1})_{n1} & \cdots & (\mathcal{L}^{n-1})_{nn} \end{pmatrix}. \tag{3.7}$$

Given the poles u_i the conjugate variables v_i can be defined from the equation

$$(\mathcal{L}(u_i) - v_i)^{\wedge}_{nk} = 0 \qquad k = 1, \ldots, n \tag{3.8}$$

and the wedge denotes the adjoint matrix.

It was shown in [1] that in the primitive domain $\mathfrak{D}$ the function $B(u)$ has exactly $N-1$ zeros u_i and $N-1$ pairs (u_i, v_i) together with the variables (Q, P), describing the motion of the centre of mass,

$$X = q_N \qquad P = \sum_{i=1}^{N} p_i \tag{3.9}$$

give the complete canonical set of new variables.

First let us examine the A_2 case (see [1]). We shall introduce another set of canonical variables. The first set $(y_i; x_i, Q; P)$ simply describes a separation of the motion of the centre of mass

$$\begin{aligned} &x_1 = q_1 - q_3 \qquad x_2 = q_2 - q_3 \qquad Q = q_3 \\ &y_1 = p_1 \qquad y_2 = p_2 \qquad P = p_1 + p_2 + p_3. \end{aligned} \tag{3.10}$$

Then we introduce two sets of canonical variables in the reduced phase space (with eliminated canonical variables (Q, P))

$$\begin{aligned} &x_+ = x_1 + x_2 \qquad x_- = x_1 - x_2 \qquad y_+ = \tfrac{1}{2}(y_1 + y_2) \qquad y_- = \tfrac{1}{2}(y_1 - y_2) \\ &u_+ = u_1 + u_2 \qquad u_- = u_1 - u_2 \qquad v_+ = \tfrac{1}{2}(v_1 + v_2) \qquad v_- = \tfrac{1}{2}(v_1 - v_2). \end{aligned} \tag{3.11}$$

The generating function $\mathcal{F}$ of the separating transformation can be written as $\mathcal{F}(v_+, u_-; x_+, x_-)$ or $\mathcal{F}(v_+, u_-; x_1, x_2)$. We prefer to use the second form, which is more convenient for a generalization to the A_3 case. This function performs the canonical transformation from $(x_{1,2}, y_{1,2})$ to $(u_\pm, v_\pm)$ such that

$$u_1 + u_2 = x_1 + x_2 \quad \mod \Gamma \tag{3.12}$$

and

$$\frac{\partial \mathcal{F}}{\partial x_1} = y_1 \qquad \frac{\partial \mathcal{F}}{\partial x_2} = y_2 \qquad \frac{\partial \mathcal{F}}{\partial v_+} = u_+ \qquad \frac{\partial \mathcal{F}}{\partial u_-} = -v_-. \tag{3.13}$$

The next trivial observation is important for a generalization to the A_3 case: the function $\mathcal{F}(v_+, u_-; x_1, x_2)$ allows the following decomposition:

$$\mathcal{F}(v_+, u_-; x_1, x_2) = v_+ x_+ + \mathrm{i}g \log \frac{\sigma(x_1)\sigma(x_2)}{\sigma(u_1)\sigma(u_2)\sigma(x_1 - x_2)} + \overline{\mathcal{F}}(u_-, x_-). \tag{3.14}$$

Here we imply that all variables in the RHS of (3.14) have to be expressed in terms of $(v_+, u_-; x_1, x_2)$ using (3.11), (3.12). Note that $\overline{\mathcal{F}}$ depends only on pairwise differences of x_i, u_i. Then we have

$$y_{1,2} = v_+ + \mathrm{i}g \left[\zeta(x_{1,2}) \mp \zeta(x_1 - x_2) - \frac{1}{2}(\zeta(u_1) + \zeta(u_2)) \right] + \overline{y}_{1,2}, \overline{y}_{1,2} = \frac{\partial \overline{\mathcal{F}}}{\partial x_{1,2}}. \tag{3.15}$$

Evaluating the determinant in (3.7) and using (3.8) we obtain that the condition $\mathfrak{B}(u) = 0$ is equivalent to

$$v_{1,2} = A_1(u_{1,2}) = A_2(u_{1,2}) \tag{3.16}$$

with

$$A_i(u) = y_i + \mathrm{i}g[\zeta(u) - \zeta(x_i) + \zeta(x_i - x_{3-i}) - \zeta(u - x_{3-i})]. \tag{3.17}$$

Using (3.15) we can rewrite (3.16) as follows:

$$\overline{y}_1 - \overline{y}_2 = 2\frac{\partial}{\partial x_-}\overline{\mathcal{F}}(u_-, x_-) = \mathrm{i}g[\zeta(u - x_2) - \zeta(u - x_1)] \qquad u = u_{1,2}. \tag{3.18}$$

It is a simple calculation to check that a solution of (3.18) which is compatible with (3.12), (3.13), (3.15), (3.16) has the following form:

$$\overline{\mathcal{F}}(u_-, x_-) = \mathrm{i}g \log \sigma\left(\frac{x_- + u_-}{2}\right)\sigma\left(\frac{x_- - u_-}{2}\right). \tag{3.19}$$

We see that the PDE (3.18) for $\overline{\mathcal{F}}$ involves a reduced number of variables and looks rather simpler than the equation (3.16) for $\mathcal{F}$. Our purpose is to obtain analogues of (3.18) for the A_3 case and try to solve them.

Again we start with a set of canonical variables $(y_i; x_i, Q; P)$, $i = 1, 2, 3$

$$\begin{array}{llll} x_1 = q_1 - q_4 & x_2 = q_2 - q_4 & x_3 = q_3 - q_4 & Q = q_4 \\ y_1 = p_1 & y_2 = p_2 & y_3 = p_3 & P = p_1 + p_2 + p_3 + p_4 \end{array} \tag{3.20}$$

and introduce in the reduced phase space canonical variables

$$\begin{array}{lll} x_+ = \dfrac{x_1 + x_2 + x_3}{3} & x' = \dfrac{2x_1 - x_2 - x_3}{3} & x'' = \dfrac{2x_2 - x_1 - x_3}{3} \\ y_+ = y_1 + y_2 + y_3 & y' = y_1 - y_3 & y'' = y_2 - y_3 \end{array} \tag{3.21}$$

and similarly a set of separated variables

$$\begin{array}{lll} u_+ = \dfrac{u_1 + u_2 + u_3}{3} & u' = \dfrac{2u_1 - u_2 - u_3}{3} & u'' = \dfrac{2u_2 - u_1 - u_3}{3} \\ v_+ = v_1 + v_2 + v_3 & v' = v_1 - v_3 & v'' = v_2 - v_3. \end{array} \tag{3.22}$$

The generating function $\mathcal{F}(v_+, u', u''; x_1, x_2, x_3)$ performs the canonical transformation from $(x_{1,2,3}, y_{1,2,3})$ to $(u_+, u', u''; v_+, v', v'')$ such that

$$u_1 + u_2 + u_3 = x_1 + x_2 + x_3 \mod \Gamma \tag{3.23}$$

and

$$\frac{\partial \mathcal{F}}{\partial v_+} = u_+ \qquad \frac{\partial \mathcal{F}}{\partial u'} = -v' \qquad \frac{\partial \mathcal{F}}{\partial u''} = -v'' \qquad \frac{\partial \mathcal{F}}{\partial x_i} = y_i \qquad i = 1, 2, 3. \tag{3.24}$$

We introduce the 'reduced' generating function $\overline{\mathcal{F}}$ by the formula

$$\mathcal{F} = v_+ x_+ + \mathrm{i}g \log \frac{\prod_{i=1}^{3} \sigma(x_i)}{\prod_{i=1}^{3} \sigma(u_i) \prod_{i<j} \sigma(x_i - x_j)} + \mathrm{i}g\overline{\mathcal{F}}. \tag{3.25}$$

Zeros of the determinant (3.7) define the separated variables u_i, $i = 1, 2, 3$. Then the conjugated variables v_i are simply rational functions of matrix elements of the Lax operator evaluated at u_i and can be found from (3.8). We want to find a convenient expression for this determinant. This calculation is quite tedious and involves complicated elliptic identities between Weierstrass functions. The easiest way to calculate $\mathfrak{B}(u)$ is to check compatibility conditions for v_i coming from (3.8) (see formulas (5.12) below). Here we shall only give the final result.

Using (3.25) let us make a change of variables

$$y_i = \tfrac{1}{3} v_+ + \mathrm{i}g \left[\zeta(x_i) - \zeta(x_i - x_j) - \zeta(x_i - x_k) - \tfrac{1}{3} \sum_{l=1}^{3} \zeta(u_l) \right] + \mathrm{i}g\overline{y}_i \tag{3.26}$$

where $\{i, j, k\}$ is a permutation of $\{1, 2, 3\}$, $\overline{y}_i = \frac{\partial}{\partial x_i}\overline{\mathcal{F}}$.

Then the determinant in (3.7) can be written as

$$\mathfrak{B}(u) = \mathrm{i}g^3 \prod_{i=1}^{3} \Phi(u, -x_i) \mathfrak{B}(r_1, r_2 | \vec{x}, u) \tag{3.27}$$

with

$$\begin{aligned} \mathfrak{B}(r_1, r_2 | \vec{x}, u) = \big\{ & \tilde{r}_1 \tilde{r}_2 (\tilde{r}_1 - \tilde{r}_2) \\ & + 2\tilde{r}_1 \tilde{r}_2 [\zeta(x_1 - u) - \zeta(x_2 - u) - \zeta(x_1 - x_2)] \\ & + \tilde{r}_1^2 [\zeta(x_1 - x_2) + \zeta(x_2 - u) - \zeta(x_1 - x_3) - \zeta(x_3 - u)] \\ & + \tilde{r}_2^2 [\zeta(x_1 - x_2) - \zeta(x_1 - u) + \zeta(x_2 - x_3) + \zeta(x_3 - u)] \big\} \end{aligned} \tag{3.28}$$

where $\vec{x} \equiv \{x_1, x_2, x_3\}$ and

$$\begin{aligned} \tilde{r}_{1,2} &= r_{1,2} + 2[\zeta(x_3 - u) - \zeta(x_{1,2} - u)] \\ r_{1,2} &= \overline{y}_{1,2} - \overline{y}_3 = \left\{ \frac{\partial}{\partial x'}, \frac{\partial}{\partial x''} \right\} \overline{\mathcal{F}}. \end{aligned} \tag{3.29}$$

From (3.28) we can see that this equation depends only on pairwise differences of x_i and u as in (3.18). Therefore, the reduced generating function $\overline{\mathcal{F}}$ depends effectively on four independent variables (say, $x_i - u_1$, $i = 1, 2, 3$ and $u_2 - u_1$). However, despite the fact that a big simplification has happened we still have a very complicated nonlinear PDE (3.28) with elliptic coefficients which is difficult to solve.

In the next sections we will show that a natural parametrization of the equation (3.28) comes from the quantum case.

4. The quantum A_3 Calogero–Moser system

For the classical A_3 CMS we have four commuting Hamiltonians [10]

$$\begin{aligned} H_1 &= \sum_{i=1}^{4} p_i \qquad H_2 = \sum_{i<j} p_i p_j - g^2 \sum_{i<j} \wp(q_i - q_j) \\ H_3 &= \sum_{i<j<k} p_i p_j p_k - g^2 \sum_{i<j} \wp(q_i - q_j)(p_k + p_l) \qquad i < j \neq k < l \\ H_4 &= p_1 p_2 p_3 p_4 - g^2 \sum_{i<j} \wp(q_i - q_j)[p_k p_l - \tfrac{1}{2} g^2 \wp(q_k - q_l)] \qquad i < j \neq k < l. \end{aligned} \tag{4.1}$$

They come from the spectral invariants of the Lax operator (2.14)

$$\det(z \cdot \mathbf{1} - \mathcal{L}(u)) = z^4 - z^3 t_1(u) + z^2 t_2(u) - z t_3(u) + t_4(u) \tag{4.2}$$

$$\begin{aligned} t_1(u) &= H_1 \\ t_2(u) &= H_2 + 6g^2 \wp(u) \\ t_3(u) &= H_3 + 3g^2 \wp(u) H_1 - 4\mathrm{i} g^3 \wp'(u) \\ t_4(u) &= H_4 - \mathrm{i} g^3 \wp'(u) H_1 + g^2 \wp(u) H_2 + g^4 [3\wp^2(u) - \wp''(u)]. \end{aligned} \tag{4.3}$$

The separated variables (v_j, u_j) satisfy the relations

$$\det(v_j \cdot \mathbf{1} - \mathcal{L}(u_j)) = v_j^4 - v_j^3 t_1(u_j) + v_j^2 t_2(u_j) - v_j t_3(u_j) + t_4(u_j) = 0. \tag{4.4}$$

Now let us consider the quantum case. We replace p_i by differentiations $p_j \to -\mathrm{i}\partial_{q_j}$ and instead of Hamiltonians (4.1) we have four commuting differential operators

$$\begin{aligned} H_1 &= -\mathrm{i} \sum_{j=1}^{4} \partial_{q_j} \qquad H_2 = -\sum_{j<k} \partial_{q_j} \partial_{q_k} - g(g-1) \sum_{j<k} \wp(q_j - q_k) \\ H_3 &= \mathrm{i} \sum_{j<k<l} \partial_{q_j} \partial_{q_k} \partial_{q_l} + \mathrm{i} g(g-1) \sum_{j<k} \wp(q_j - q_k)(\partial_{q_l} + \partial_{q_m}) \\ H_4 &= \partial_{q_1} \partial_{q_2} \partial_{q_3} \partial_{q_4} + g(g-1) \sum_{j<k} \wp(q_j - q_k) \left[\partial_{q_l} \partial_{q_m} + \frac{g(g-1)}{2} \wp(q_l - q_m) \right] \end{aligned} \tag{4.5}$$

where $j < k \neq l < m$.

As explained in [2] for the A_2 case (see also [8] for a trigonometric case) the idea is to construct the linear operator K which intertwines $\{q_i\}$ and $\{u_i; Q\}$ representations.

Namely, we are looking for the kernel $K(\vec{u}, Q; \vec{q})$, $\vec{u} = \{u_1, u_2, u_3\}$, $\vec{q} = \{q_1, q_2, q_3, q_4\}$ of the operator K such that

$$K(\vec{u}, Q; \vec{q}) = \delta(Q - q_4) \tilde{K}(\vec{u}; \vec{x}) \tag{4.6}$$

where the variables u_i, x_i are defined by (3.20)–(3.22). The spectral determinant (4.4) is replaced by the following differential equation for the kernel K:

$$\begin{aligned} [\partial_{u_j}^4 &- \mathrm{i} H_1^* \partial_{u_j}^3 - [H_2^* + 6g(g-1)\wp(u_j)] \partial_{u_j}^2 \\ &+ [\mathrm{i} H_3^* + 3\mathrm{i} g(g-1) H_1^* \wp(u_j) + 4g(g-1)(g-2)\wp'(u_j)] \partial_{u_j} \\ &+ H_4^* + g(g-1) H_2^* \wp(u_j) - \mathrm{i} g(g-1)(g-2) H_1^* \wp'(u_j) \\ &+ 3g^2(g-1)^2 \wp^2(u_j) - g(g-1)(g^2 - 3g + 3)\wp''(u_j)] K = 0 \end{aligned} \tag{4.7}$$

where H_i^* is the Lagrange adjoint of H_i

$$\int \phi(\vec{q})(H\psi)(\vec{q})\, \mathrm{d}\vec{q} = \int (H^*\phi)(\vec{q}) \psi(\vec{q})\, \mathrm{d}\vec{q} \tag{4.8}$$

and the condition $P = -\mathrm{i}\partial_Q$ is replaced by

$$[-\mathrm{i}\partial_Q - H_1^*]K = 0 \tag{4.9}$$

which is trivially satisfied because of (4.6).

One of the possible ways to fix coefficients in (4.7) is to look at two different limits: the classical one (when $g \to \infty$ and we should have (4.4)) and the trigonometric limit $\wp(x) \to \csc(x)^2$, $\zeta(x) \to \cot(x)$, where the version of (4.4) has been conjectured in [8]. These two limits fix coefficients in (4.7) uniquely.

Let us assume that $\Psi(\vec{q})$ is an eigenfunction of H_i, $i = 1, 2, 3, 4$ and consider the integral transform

$$\tilde{\Psi}(\vec{u}, Q) = \int \mathrm{d}\vec{q}\; K(\vec{u}, Q; \vec{q})\Psi(\vec{q}). \tag{4.10}$$

Now we demand that the function $\tilde{\Psi}(\vec{u}, Q)$ should satisfy the separated equations

$$[-\mathrm{i}\partial_Q - h_1]\tilde{\Psi}(\vec{u}, Q) = 0 \qquad \mathfrak{D}_{u_j}\tilde{\Psi}(\vec{u}, Q) = 0 \tag{4.11}$$

where

$$\begin{aligned}\mathfrak{D}_y = \partial_y^4 - \mathrm{i}h_1\partial_y^3 - [h_2 + 6g(g-1)\wp(y)]\partial_y^2 \\ +[\mathrm{i}h_3 + 3\mathrm{i}g(g-1)h_1\wp(y) + 4g(g-1)(g-2)\wp'(y)]\partial_y \\ +h_4 + g(g-1)h_2\wp(y) - \mathrm{i}g(g-1)(g-2)h_1\wp'(y) \\ +3g^2(g-1)^2\wp^2(y) - g(g-1)(g^2-3g+3)\wp''(y)\end{aligned} \tag{4.12}$$

and h_i are the eigenvalues of H_i corresponding to the eigenfunction $\Psi(\vec{q})$.

We are not going to discuss in this paper the question of correct boundary conditions for the operator K and differential equation (4.12). We only make an assumption that the boundary can be chosen in such a way that it does not contribute to the result while integrating by parts using (4.7), (4.8). Unlike the A_2 case a correct choice of boundary conditions for (4.10) appears to be quite a complicated problem even in the trigonometric limit and we will address this problem in a separate paper.

Our purpose is to solve exactly the differential equation (4.7) for the kernel K. Substituting adjoints of Hamiltonians H_i (4.5) into (4.7), using a factorization (4.6) of the kernel K and the change of variables (3.20) we come to the following equation:

$$\mathfrak{D}_{u_j}^{(1)}(u_j; \vec{x})\tilde{K}(\vec{u}; \vec{x})\,\partial_Q + \mathfrak{D}_{u_j}^{(0)}(u_j; \vec{x})\tilde{K}(\vec{u}; \vec{x}) = 0 \tag{4.13}$$

where $\mathfrak{D}_{u_j}^{(1)}(u_j; \vec{x})$ and $\mathfrak{D}_{u_j}^{(0)}(u_j; \vec{x})$ are the third- and fourth-order differential operators in u_j, respectively. The kernel $\tilde{K}(\vec{u}; \vec{x})$ should solve both of them.

Now guided by the A_3 classical case let us make a substitution

$$\tilde{K}(\vec{u}; \vec{x}) = \left\{\frac{\prod_{i=1}^3 \frac{\sigma(x_i)}{\sigma(u_i)}}{\prod_{i<j}\sigma(x_i - x_j)}\right\}^{g-1} \tilde{L}(\vec{u}; \vec{x}) \tag{4.14}$$

and assume that the reduced kernel $\tilde{L}(\vec{u}; \vec{x})$ has the following invariance:

Conjecture 4.1.

$$\tilde{L}(u_1 + \lambda, u_2 + \lambda, u_3 + \lambda; x_1 + \lambda, x_2 + \lambda, x_3 + \lambda) = \tilde{L}(\vec{u}; \vec{x}) \qquad \forall\lambda \in \mathfrak{D}. \tag{4.15}$$

It appears that both equations in (4.13) are compatible with (4.15) provided that $\tilde{L}(\vec{u}; \vec{x})$ satisfies the following system of linear PDEs with elliptic coefficients:

$$\begin{aligned}\big\{(g-1)^2[\wp(x_\alpha - x_\beta) - \wp(x_\gamma - u_i) - \zeta^2(x_\alpha - x_\beta) + \zeta^2(x_\gamma - u_i)] \\ +(g-1)[\zeta(x_\alpha - x_\beta)(\partial_{x_\alpha} - \partial_{x_\beta}) + \zeta(x_\gamma - u_i)(2\partial_{u_i} + \partial_{x_\alpha} + \partial_{x_\beta})] \\ +(\partial_{u_i} + \partial_{x_\alpha})(\partial_{u_i} + \partial_{x_\beta})\big\}\tilde{L}(\vec{u}; \vec{x}) = 0 \qquad i, \alpha < \beta = 1, 2, 3.\end{aligned} \tag{4.16}$$

Again the differential operator in (4.16) of the second order is considerably simpler than differential operators in (4.13). The statement that the kernel $\tilde{K}$ (4.14) with $\tilde{L}$ satisfying (4.16) will solve (4.13) can be proved by direct calculations (very lengthy). In fact, using the conjecture 4.1 the equations (4.16) can be obtained only from the equation $\mathfrak{D}^{(1)}_{u_j}(u_j;\vec{x})\tilde{K}(\vec{u};\vec{x}) = 0$. Then the second equation $\mathfrak{D}^{(0)}_{u_j}(u_j;\vec{x})\tilde{K}(\vec{u};\vec{x}) = 0$ is valid automatically.

We strongly believe that $\tilde{K}$ with the factorization (4.14) and $\tilde{L}$ satisfying (4.16) is the only sensible solution to (4.13). However, it would be very interesting to find other solutions to (4.13) which are not of the form (4.14).

Now we will solve the system (4.16) for the kernel $\tilde{L}(\vec{u};\vec{x})$.

Theorem 4.2. *The kernel $\tilde{L}(\vec{u};\vec{x})$ admits further factorization*

$$\tilde{L}(\vec{u};\vec{x}) = \delta(u_+ - x_+)L(\vec{t},s) \tag{4.17}$$

$$\vec{t} = \{t_1,t_2,t_3\} \qquad t_i = x_i - u_1 \qquad s = u'' - u' = u_2 - u_1 \tag{4.18}$$

with u_+, u', u'' defined in (3.22).

Proof. Consider (4.16) for $i = 1,2,3$ and rewrite it terms of variables $t_i = x_i - u_1$, $s = u_2 - u_1$, u_1 and $\Delta = u_+ - x_+$. Using the conjecture 4.1 and comparing mixed derivatives of $\tilde{L}$ one can show that (4.16) is compatible only if

$$\tilde{L}(u_1,s;t_1,t_2,t_3;\Delta) \sim \delta(\Delta). \tag{4.19}$$

□

Now introduce differential operators:

$$\begin{aligned}\mathfrak{D}_{\alpha\beta} \equiv{}& \partial_{t_\alpha}\partial_{t_\beta} + (g-1)[\zeta(t_\alpha - t_\beta)(\partial_{t_\alpha} - \partial_{t_\beta})\\ &+\zeta(s - t_\alpha - t_\beta)(\partial_{t_\alpha} + \partial_{t_\beta})] + (g-1)^2\\ &\times[\wp(t_\alpha - t_\beta) - \wp(s - t_\alpha - t_\beta) - \zeta^2(t_\alpha - t_\beta) + \zeta^2(s - t_\alpha - t_\beta)]\end{aligned} \tag{4.20}$$

$$\begin{aligned}\mathfrak{D}'_{\alpha\beta} \equiv{}& (\partial_{t_\alpha} + \partial_s)(\partial_{t_\beta} + \partial_s)\\ &+(g-1)[\zeta(t_\alpha - t_\beta)(\partial_{t_\alpha} - \partial_{t_\beta}) + \zeta(t_\gamma - s)(\partial_{t_\alpha} + \partial_{t_\beta} + 2\partial_s)\\ &+(g-1)^2[\wp(t_\alpha - t_\beta) - \wp(t_\gamma - s) - \zeta^2(t_\alpha - t_\beta) + \zeta^2(t_\gamma - s)]\end{aligned} \tag{4.21}$$

$$\begin{aligned}\mathfrak{D}''_{\alpha\beta} \equiv{}& (\partial_{t_\alpha} + \partial_{t_\gamma} + \partial_s)(\partial_{t_\beta} + \partial_{t_\gamma} + \partial_s)\\ &+(g-1)[\zeta(t_\alpha - t_\beta)(\partial_{t_\alpha} - \partial_{t_\beta}) - \zeta(t_\gamma)(\partial_{t_\alpha} + \partial_{t_\beta} + 2\partial_{t_\gamma} + 2\partial_s)]\\ &+(g-1)^2[\wp(t_\alpha - t_\beta) - \wp(t_\gamma) - \zeta^2(t_\alpha - t_\beta) + \zeta^2(t_\gamma)]\end{aligned} \tag{4.22}$$

where α, β, γ is a permutation of $\{1,2,3\}$. Then the system (4.16) for the kernel $\tilde{L}(\vec{u};\vec{x})$ is equivalent to the following system of equations for $L(\vec{t},s)$:

$$\mathfrak{D}_{\alpha\beta}L(\vec{t},s) = 0 \qquad \mathfrak{D}'_{\alpha\beta}L(\vec{t},s) = 0 \qquad \mathfrak{D}''_{\alpha\beta}L(\vec{t},s) = 0. \tag{4.23}$$

The following theorem is an elliptic generalization of the result given in [12].

Theorem 4.3. *A solution for the system (4.23) is given by the following expression:*

$$L(\vec{t},s) = \oint_C \mathrm{d}z\, \tau(\vec{t},s|z) \tag{4.24}$$

where

$$\tau(\vec{t},s|z) = \varkappa(\vec{t},s|z)^{g-1} \tag{4.25}$$

$$\varkappa(\vec{t},s|z) = \frac{\sigma(z)\sigma(z+s)}{\sigma^2(2z+s)}\prod_{i=1}^{3}\sigma(z+t_i)\sigma(z+s-t_i) \tag{4.26}$$

and the contour C is closed on the Riemann surface of the integrand.

Proof. The proof of the theorem is based on three elliptic identities:

$$\mathfrak{D}_{\alpha\beta}\big[\tau(\vec{t}, s|z)\big] = 0 \tag{4.27}$$

$$\mathfrak{D}'_{\alpha\beta}\big[\tau(\vec{t}, s|z)\big] = (g-1)\frac{\partial}{\partial z}\left[\frac{\sigma(z)\sigma(z+t_\gamma)\sigma(2z+2s-t_\gamma)\tau(\vec{t}, s|z)}{\sigma(z+s-t_\gamma)\sigma(z+s)\sigma(2z+s)\sigma(t_\gamma-s)}\right] \tag{4.28}$$

and

$$\mathfrak{D}''_{\alpha\beta}\big[\tau(\vec{t}, s|z)\big] = (g-1)\frac{\partial}{\partial z}\left[\frac{\sigma(z)\sigma(t_\gamma-z-s)\sigma(2z+s+t_\gamma)\tau(\vec{t}, s|z)}{\sigma(z+s)\sigma(2z+s)\sigma(t_\gamma)\sigma(z+t_\gamma)}\right]. \tag{4.29}$$

Formulas (4.27)–(4.29) can be proved either by using (2.8)–(2.11) or checking that a difference of LHS and RHS are elliptic functions with no poles.

These identities show that under the action of $\mathcal{D}_{\alpha\beta}, \mathcal{D}'_{\alpha\beta}, \mathcal{D}''_{\alpha\beta}$ the integral in (4.24) becomes a total derivative of the function with the same singularities as the function $\tau(\vec{t}, s|z)$ in (4.25). $\square$

A natural question arises: do (4.24)–(4.26) describe a general solution to the system (4.23)? In the trigonometric limit the answer is positive and changing the contour $\mathcal{C}$ we can produce the whole basis of linearly independent solutions [12]. It is likely that this statement can be generalized to the elliptic case as well.

In fact, all we have proved is that if we choose integration contours to be some curves in (4.24) and (4.10) closed on the Riemann surface of the integrands, then the equations (4.11) should be valid. Of course, it does not guarantee that the function (4.10) will split into the product of functions depending on Q and u_i separately. However, the asymptotics of the integral (4.24) in the classical limit $g \to \infty$ provides a natural parametrization for (3.28).

So let us consider the limit $g \to \infty$. Then we have to calculate the asymptotic behaviour of the reduced kernel $L(\vec{t}, s)$ when $g \to \infty$. It is clear that, in general, this asymptotics is a multi-valued function of (t_1, t_2, t_3, s). Due to a special form (4.25) of $\tau(\vec{t}, s|z)$ we can use the steepest-descent method to obtain that

$$L(\vec{t}, s)|_{g\to\infty} \backsimeq \exp(g \log \mathfrak{F}(\vec{t}, s)) \tag{4.30}$$

where

$$\mathfrak{F}(\vec{t}, s) = \varkappa(\vec{t}, s|z^*) \tag{4.31}$$

with

$$\frac{\partial}{\partial z}\varkappa(\vec{t}, s|z^*)|_{z=z^*} = 0. \tag{4.32}$$

We can rewrite the equation (4.32) for z^* as

$$\zeta(z^*) + \zeta(z^*+s) + \sum_{i=1}^{3}[\zeta(t_i+z^*) + \zeta(s-t_i+z^*)] = 4\zeta(s+2z^*). \tag{4.33}$$

This equation defines the stationary phase point z^* at which the function (4.31) has to be evaluated.

5. The A_3 generating function

We shall start with the following lemma, which provides the main technical tool for the construction of the A_3 generating function.

Lemma 5.1. *The function $\varkappa(\vec{t}, s|z)$ satisfies the following PDE with elliptic coefficients:*

$$\mathfrak{B}\left(\frac{\partial}{\partial x'}\log\varkappa(\vec{t}, s|z), \frac{\partial}{\partial x''}\log\varkappa(\vec{t}, s|z)|\vec{t}, v\right)$$
$$= \frac{\partial}{\partial z}\varkappa(\vec{t}, s|z)\frac{\sigma(\sum_{i=1}^{3} t_i - s - v)\sigma(z)^2\sigma(z+s)^2\prod_{i<j}\sigma(t_i - t_j)}{\varkappa(\vec{t}, s|z)^2\sigma(2z+s)\prod_{i=1}^{3}\sigma(t_i - v)} \tag{5.1}$$

where $v = \{0, s, t_1 + t_2 + t_3 - s\}$, $t_i = x_i - u_1$, $s = u_2 - u_1$ and the function $\mathfrak{B}(r_1, r_2; \vec{x}, u)$ is defined by relations (3.28), (3.29).

The proof of the lemma is straightforward, but technically complicated. It is instructive to start with the case $v = t_1 + t_2 + t_3 - s$, when the RHS in (5.1) is zero. Then the LHS is some combination of Weierstrass ζ functions which is zero. The first two cases $v = \{0, s\}$ are some of the most complicated elliptic identities in this paper. They can be proved in several steps using identities similar to (5.12) and (5.14)–(5.16) (see below).

Now we are ready to formulate the main result of this paper.

Theorem 5.2. *The A_3 generating function $\mathcal{F}(v_+, u', u''; x_1, x_2, x_3)$ performing the canonical transformation from $(x_{1,2,3}, y_{1,2,3})$ to $(u_+, u', u''; v_+, v', v'')$ is given by*

$$\mathcal{F} = v_+x_+ + \mathrm{i}g\log\frac{\prod_{i=1}^{3}\sigma(x_i)}{\prod_{i=1}^{3}\sigma(u_i)\prod_{i<j}\sigma(x_i - x_j)} + \mathrm{i}g\overline{\mathcal{F}} \tag{5.2}$$

with $\overline{\mathcal{F}}(\vec{t}, s)$

$$\overline{\mathcal{F}}(\vec{t}, s) = \log\mathfrak{F}(\vec{t}, s) \tag{5.3}$$

where $\mathfrak{F}(\vec{t}, s)$ is defined by (4.31)–(4.33), the variables $\vec{t}, s$ by (4.18) and all variables $u_i, u_+, u', u'', v_i, v_+, v', v'', x_i, y_i$ by (3.20)–(3.22).

Proof. The proof proceeds in two steps. First of all we have to check that with the generating function (5.2) the equation $\mathfrak{B}(u) = 0$ has three roots u_1, u_2, u_3 in the primitive domain $\mathfrak{D}$. Now using lemma 5.1 and definitions (3.20)–(3.29) it is easy to see that three roots u_1, u_2, u_3 correspond exactly to the cases $v = \{0, s, t_1 + t_2 + t_3 - s\}$ of lemma 5.1. So choosing z^* such that $\frac{\partial}{\partial z}\varkappa(\vec{t}, s|z)|_{z=z^*} = 0$ we obtain the solution to $\mathfrak{B}(u) = 0$.

The next step is to show that the conjugated variables v_1, v_2, v_3 (or v_+, v', v'') defined by (3.8) are compatible with (5.2).

Let us denote as v_1^*, v_2^*, v_3^* the conjugated variables obtained from the equations

$$\frac{\partial}{\partial u'}\mathcal{F} = -v_1^* + v_3^* \qquad \frac{\partial}{\partial u''}\mathcal{F} = -v_2^* + v_3^* \tag{5.4}$$

and

$$y_+ = \sum_{i=1}^{3} v_i^* + \mathrm{i}g\left(\sum_{i=1}^{3}[\zeta(x_i) - \zeta(u_i)] + \sum_{i=1}^{3}\overline{y}_i\right) \tag{5.5}$$

where we simply used (3.26) and

$$\overline{y}_i = \frac{\partial}{\partial x_i}\overline{\mathcal{F}} = \zeta(z + x_i - u_1) - \zeta(z + u_2 - x_i) - \frac{1}{3}\sum_{i=1}^{3}[\zeta(z + x_i - u_1) - \zeta(z + u_2 - x_i)] \tag{5.6}$$

from the formula for $\overline{\mathcal{F}} = \log\varkappa(\vec{t}, s|z)$. Note that $\sum_{i=1}^{3}\overline{y}_i = 0$ simply reflects the fact that $\overline{\mathcal{F}}$ depends only on four variables u', u'', x', x''.

Substituting (5.2) into (5.4) and using (5.5), (5.6) we obtain the following expressions for v_i^*:

$$v_1^* = \tfrac{1}{3}y_+ + \mathrm{i}g\Big[\zeta(u_1) + \zeta(z+u_2-u_1) - 2\zeta(2z+u_2-u_1) + \\ +\tfrac{1}{3}\sum_{i=1}^{3}[\zeta(z+u_2-x_i) + 2\zeta(z+x_i-u_1) - \zeta(x_i)]\Big] \tag{5.7}$$

$$v_2^* = \tfrac{1}{3}y_+ + \mathrm{i}g\Big[\zeta(u_2) - \zeta(z+u_2-u_1) + 2\zeta(2z+u_2-u_1) \\ +\tfrac{1}{3}\sum_{i=1}^{3}[-2\zeta(z+u_2-x_i) - \zeta(z+x_i-u_1) - \zeta(x_i)]\Big] \tag{5.8}$$

$$v_3^* = \tfrac{1}{3}y_+ + \mathrm{i}g\Big[\zeta(u_3) + \tfrac{1}{3}\sum_{i=1}^{3}[\zeta(z+u_2-x_i) - \zeta(z+x_i-u_1) - \zeta(x_i)]\Big]. \tag{5.9}$$

Now we check that these expressions are compatible with (3.8). The conditions (3.8) were analysed in [1] where many different expressions for v_i were obtained. All these expressions are equivalent provided that $\mathfrak{B}(u_i) = 0$. Let us introduce matrices [1]

$$\mathcal{L}^{(p)}(u) = \mathcal{L}(u)[\mathrm{tr}\,\mathcal{L}^{(p-1)}(u)] - (p-1)\mathcal{L}^{(p-1)}(u)\mathcal{L}(u) \qquad \mathcal{L}^{(1)}(u) = \mathcal{L}(u). \tag{5.10}$$

Then from formula (3.23) of [1] with $n = 4$, $i = 1$, $j = 3$, $k = \alpha$, $\alpha = 1, 2$ we have

$$v_i = \frac{\mathcal{L}^{(1)}_{4\alpha}(u_i)\mathcal{L}^{(3)}_{43}(u_i) - \mathcal{L}^{(1)}_{43}(u_i)\mathcal{L}^{(3)}_{4\alpha}(u_i)}{\mathcal{L}^{(1)}_{4\alpha}(u_i)\mathcal{L}^{(2)}_{43}(u_i) - \mathcal{L}^{(1)}_{43}(u_i)\mathcal{L}^{(2)}_{4\alpha}(u_i)}. \tag{5.11}$$

Now using the definition (2.14) of the Lax operator $\mathcal{L}(u)$ and the addition theorem (2.11) one can rewrite (5.11) in the following form:

$$v_i = y_\alpha + \mathrm{i}g\Bigg[\zeta(x_3-u_i) + \zeta(u_i) - \zeta(x_\alpha) + \zeta(x_\alpha - x_3) + \frac{\tilde{r}_\alpha(u_i)}{\tilde{r}_{3-\alpha}(u_i)} \\ \times\big[\zeta(u_i-x_3) - \zeta(u_i-x_{3-\alpha}) + \zeta(x_\alpha - x_{3-\alpha}) - \zeta(x_\alpha - x_3)\big]\Bigg] \tag{5.12}$$

$$\tilde{r}_\alpha(u_i) = \overline{y}_\alpha - \overline{y}_3 + 2\zeta(u_i - x_\alpha) - 2\zeta(u_i - x_3) \qquad \alpha = 1, 2. \tag{5.13}$$

It is easy to see that two expressions (5.12) for $\alpha = 1, 2$ are equivalent exactly when $\mathfrak{B}(u_i) = 0$. Let us use (5.12) with $\alpha = 1$. The following three elliptic identities can be proved using (2.11). In fact, they are very useful in the proof of lemma 5.1. We shall put them in the form convenient for our purposes, namely,

$$v_1 - v_1^* = \mathrm{i}g\frac{1}{\tilde{r}_2(u_1)}\frac{\partial}{\partial z}\log\varkappa(\vec{t}, s|z) \\ \times[\zeta(x_2-u_1) - \zeta(x_3-u_1) - \zeta(z+x_2-u_1) + \zeta(z+x_3-u_1)] \tag{5.14}$$

$$v_2 - v_2^* = \mathrm{i}g\frac{1}{\tilde{r}_2(u_2)}\frac{\partial}{\partial z}\log\varkappa(\vec{t}, s|z) \\ \times[\zeta(u_2-x_2) - \zeta(u_2-x_3) - \zeta(u_2-x_2+z) + \zeta(u_2-x_3+z)] \tag{5.15}$$

$$v_3 - v_3^* = 0. \tag{5.16}$$

These elliptic identities show that if we choose again z to be z^* such that $\frac{\partial}{\partial z}\log\varkappa(\vec{t}, s|z)|_{z=z^*} = 0$, then the generating function $\mathcal{F}(v_+, u', u''; x_1, x_2, x_3)$ satisfies

$$\mathrm{d}(\mathcal{F} - v_+u_+) = y_1\,\mathrm{d}x_1 + y_2\,\mathrm{d}x_2 + y_3\,\mathrm{d}x_3 - v_+\mathrm{d}u_+ - v'\,\mathrm{d}u' - v''\,\mathrm{d}u''. \tag{5.17}$$

This identity proves that the transformation from $(x_1, x_2, x_3, y_1, y_2, y_3)$ to $(u_+, u', u''; v_+, v', v'')$ is canonical [13]. □

6. Conclusion

In this paper we have constructed the generating function of the canonical separating transform for the A_3 classical CMS. This function appears to be multi-valued. The approach we used originated from the quantum version of the model. In fact, the purpose of this paper was to show that the conjectured quantum separating operator produces the correct asymptotics in the classical limit. It gives us confidence that we have obtained the correct expression for the quantum kernel. However, the problem of correct boundary conditions looks complicated because of quite nontrivial monodromy properties of this operator. We also think that it is straightforward to generalize the results of this paper for the classical Ruijsenaars system in line with [1]. We hope to address these problems in further publications.

Of course, a generalization of these results even for the classical A_n $(n > 4)$ case would be of great interest. The classical A_3 Calogero–Moser model appears to be the first case where the generating function is a function 'living' on some complicated Riemann surface. However, we believe that a consideration of the classical case can give a key to constructing the A_n quantum separating operator.

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